# Summer of Science

## Game Theory

End-Term Report

Vansh Kapoor
Mentor: Aryan

# Game Theory

Game Theory is a mathematical concept that models the interaction between multiple agents.

There are 3 critical factors in modeling a Game:

- **Players**: They are the decision-makers. Eg: individuals, Government, Companies
- **Actions:** They define what the players are capable of doing. Eg: all legal moves for a knight while playing chess, deciding when to sell a stock, deciding what card to play next in a game of UNO, etc.
- **Payoffs**: They are what motivate players. They take into account the profits/losses a player receives.

Standard Representation:

- **Normal Form:** It naturally captures, but is not limited to, the game where agents play simultaneously and receive immediate payoffs. It is also known as **Matrix Form** as it can be easily represented in the form of a matrix.

- **Extensive Form:** It naturally captures sequentially played games and keeps track of what each player knows when he or she makes an action.

## Nash Equilibrium and Pareto Optimality

A game is said to have attained Nash equilibrium if no player has the incentive to deviate from their action. In other words, each player's action maximizes his or her payoff given the actions of the others.

So the action played by each player is the best response in terms of payoff given the action profile of the rest of the players.

An outcome **A** is said to Pareto dominate another possible outcome, let's say **B** of a game if its payoff is at least as good as **B** for every agent and offers a payoff higher than that of B at least for one agent.

An outcome is Pareto-optimal if there is no other outcome that Pareto-dominates it.

## Mixed Strategies

- **Pure Strategy:** Only a single action is played with a positive probability

- **Mixed Strategy:** More than one action is played with positive probability (They are called the support of the mixed strategy)

The following theorem holds for mixed strategies:

Theorem: Every finite game has a Nash equilibrium

Procedure for finding Nash equilibrium with players playing mixed strategies

1. An important point to note is that each player tries to make the other player indifferent to choosing his possible actions by defining an appropriate probability distribution over his actions.
2. This is because player 1 best responds with a mixed strategy to maximize his payoff, player 2 must make him indifferent between the actions that he can play

## Strictly Dominated Strategies and their iterated removal

A strategy **S1** is said to be strictly dominated by another strategy **S2** for a player if the payoffs received by the player playing **S1** are strictly less than the payoffs he would receive if he played **S2** for all possible actions played by other players.

Iterated removal of strictly dominated strategies preserves Nash equilibria, which can be employed as a preliminary step before calculating an equilibrium. Certain games, known as dominance solvable games, can be effectively solved using this approach.

## Maxmin and Minmax Strategies

The maxmin strategy for player i is the strategy that maximizes their **worst-case payoff** when all other players (referred to as -i) play the strategies that inflict the most harm on player i. The maxmin value, also known as the **safety level**, represents the minimum payoff that can be guaranteed by employing a maxmin strategy for player i in the game.

The maxmin value for player 1 is called the **value of the game**.

In a 2-player game, **player i's** minmax strategy against **player -i** is the strategy that seeks to minimize the best-case payoff for **player -i**. The minmax value for player i against player -i represents the payoff associated with this strategy.

Why play a maxmin strategy?

- A conservative agent would like to play risk-free and maximize his worst-case payoff
- A paranoid agent who believes everyone is against him

Why play a minmax strategy?

- To cause as maximum harm as possible to other players

Min-Max Theorem:

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

The set of maxmin strategies for both players is equivalent to the set of minmax strategies. Any strategy profile that represents a maxmin strategy (or minmax strategy) constitutes a Nash equilibrium! Additionally, these maxmin/minmax strategy profiles encompass all Nash equilibria in the game. Consequently, all Nash equilibria share the same payoff vector, specifically the ones where player 1 obtains the value of the game.

## Imperfect Information Extensive form

In the normal form representation of a game, there is no consideration for the sequence or timing of players' actions. However, an alternative representation called the extensive form is available, which explicitly accounts for the temporal structure and order of actions in the game.

There are two types of Extensive forms:

- **Perfect Information Extensive forms**
- **Imperfect Information Extensive forms**

There are multiple sub-concepts in Imperfect information games like **sub-game perfect equilibria,** but an important point to note is that an extensive form could be represented in Normal form, but vise-versa isn't true. Therefore all the concepts of Normal form games still hold here!

## Bayesian Games

Currently, our focus lies on exploring games involving incomplete information, wherein at least one player possesses undisclosed knowledge about the game, unbeknownst to other players. These games fall under the category of Bayesian Games.

In essence, a game with incomplete information refers to a scenario where, during a player's turn to act, at least one participant holds private information about the game, known as the player's "type" right before making their move.

When delving into the study of such games, we make the following assumptions:

1. Each player (denoted by i) possesses a comprehensive understanding of the game's structure, as defined earlier.
2. Every player (i) is aware of their own type ($\theta_i \in \Theta_i$). This type is obtained through various signals, and each element within the type set represents a

condensed summary of the information derived from those signals.

3. The aforementioned facts are common knowledge shared among all the players in the group (denoted by N).
4. The precise type of a player remains indeterminable to other participants. However, they do have a probabilistic estimation of what this type could be. These conditional probabilities are described by belief functions (pi), which are also common knowledge among all the players.

## Seltan Game

This is a representation of Bayesian games that enables a Bayesian game to be transformed to a strategic form game (with complete information).

The concept behind creating a Selten game involves introducing type agents. In this modified version, each player from the original Bayesian game is substituted with a group of type agents. Notably, the number of type agents for a specific player corresponds precisely to the number of types present in that player's type set.

Furthermore, it is reasonable to assume that the type sets of the players have no overlap, ensuring that each type agent belongs exclusively to one player and represents a unique type from their respective type set.

## Bayesian Nash Equilibrium

A pure strategy Bayesian Nash equilibrium in a Bayesian game can be defined in a natural way as a pure strategy Nash equilibrium of the **equivalent Selten game.**

Bayesian Nash equilibrium generalizes the traditional concept of Nash equilibrium, which is applicable to games with complete information. In Bayesian games, players must consider not only their opponents' strategies but also their private information and how that information affects their decisions.

Finding Bayesian Nash equilibria can be more **challenging** than regular Nash equilibria due to the added complexity of incomplete information. However, it provides a **powerful framework** for modeling real-world scenarios where players have different levels of knowledge and information about each other, making it a valuable tool in the analysis of strategic interactions in various fields, such as economics, political science, and computer science.

# Mechanism Design

Mechanism design is an intricate and strategic art that involves crafting games in a manner that encourages and **elicits desirable behavior** among its participants. It can be conceptualized as the reverse engineering of games or, more aptly put, as the art of carefully devising the rules and framework of a game to achieve **specific and desired outcomes.**

At its core, mechanism design seeks to create institutions or protocols that not only meet certain predetermined objectives but also take into account the **strategic nature** of individual agents who interact within these structures, wherein these agents may hold private information that bears significance to the decision-making process.

The **beauty of mechanism design** lies in its ability to align the interests of **rational and self-interested agents** with the overall system-wide goals. By skillfully devising the rules and incentives of the game, mechanism design fosters an environment where participants are encouraged to act in ways that lead to collectively beneficial outcomes. It takes into consideration the strategic interactions between agents and aims to mitigate potential conflicts of interest that could arise during the decision-making process.

Mechanisms, in this context, refer to the rules and procedures that govern how the game is played. They induce a game among the strategic agents to realize a broader social choice function or system-wide objective. These mechanisms can be classified into two main categories: **direct and indirect mechanisms**. Direct mechanisms involve straightforward and explicit interactions between the players and the institution, where participants' actions directly impact the outcome of the game. On the other hand, indirect mechanisms work more subtly, often through intermediary steps or mechanisms, to elicit the desired behavior from the participants.

A key challenge in mechanism design is addressing the informational asymmetry among the players. Since individual agents may possess private information that can influence their decisions, crafting mechanisms that incentivize truthful revelation becomes crucial. Moreover, the design should anticipate strategic behavior and take into account potential manipulations or gaming of the system by **self-interested participants.**

In essence, mechanism design is a powerful tool for shaping the outcomes of interactions among rational agents in a strategic environment. By thoughtfully constructing the rules of the game, mechanism design aims to steer the collective actions of individuals towards socially desirable results, thereby promoting **cooperation and efficient decision-making** in various real-world scenarios.